Trigonometry via the Complex Exponential

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The trigonometric functions

The trigonometric or circular functions include the following:

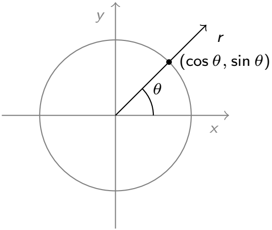
$$\sin\theta, \quad \cos\theta, \quad \tan\theta, \quad \csc\theta, \quad \sec\theta, \quad \cot\theta.$$

We define tangent, cosecant, secant, and cotangent all in terms of sine and cosine, so it suffices to focus on those two functions.

The traditional definition of sine and cosine

Here is how the traditional definition goes:

In $\mathbf{R}^2$, let r be a ray emanating from the origin, whose angle relative to the positive x axis is θ. Orient so that $\theta > 0$ for counterclockwise rotations and $\theta < 0$ for clockwise rotations. This ray intersects the unit circle at a unique point with Cartesian coordinates $(\cos\theta, \sin\theta)$.



The situation

The definition above relies on *geometric intution*. In a similar way, so do the proofs of the trigonometric identities that use this definition.

I prefer a different style, one that leverages the tools of algebra and analysis. Symbol manipulations are easy to follow step by step, and the tools of analysis make precise what geometry leaves to the imagination.

Anyway, the trigonometric identities *feel very algebraic*: they're a set of exact rules that apply to certain kinds of functions. Ideally, their proofs should be algebraic in flavor.

A more systematic solution

Enter the complex exponential,

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \forall z \in \mathbf{C}.$$

Rudin called it “the most important function in mathematics.”

It has an infinite radius of convergence via the ratio test, and has a derivative of every order, also defined on all of $\mathbf{C}$: namely, itself.

New definition

We thus define sine and cosine as follows:

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z := \frac{e^{iz} + e^{-iz}}{2}.$$

Since both functions are linear combinations of e^{iz} and e^{-iz}, we see that sine and cosine (and all derivatives thereof) are defined on all of $\mathbf{C}$ as well.

Power series for sine and cosine

One immediate consequence of the definitions is that sine and cosine can also be expressed as power series:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}.$$

Euler's formula

Another immediate consequence of the definitions is **Euler's formula**,

$$e^{iz} = \cos z + i \sin z.$$

Lie group homomorphism

The complex exponential function is an homomorphism of the Lie groups

$$(\mathbf{C}, +) \quad \text{and} \quad (\mathbf{C}^\times, \times).$$

This can be shown by applying the binomial theorem to the power series definition as follows:

$$\begin{aligned}
e^z e^w &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^n w^m}{n!\, m!} \\
&= \sum_{\ell=0}^{\infty}\sum_{n=0}^{\ell} \frac{z^n w^{\ell-n}}{n!(\ell-n)!} = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\sum_{n=0}^{\ell} \binom{\ell}{n} z^n w^{\ell-n} \\
&= \sum_{\ell=0}^{\infty} \frac{(z+w)^\ell}{\ell!} = e^{z+w}.
\end{aligned}$$

Using the homomorphism

The property $e^{z+w} = e^z e^w$ is fundamental.

Having established it, we may now prove the trigonometric identities as algebraic corollaries of this property.

Angle sum formulae

To prove these, we restrict to the real line so that $\cos\theta$ is the real part of the unit complex number $e^{i\theta}$, and $\sin\theta$ is the imaginary part.

Computing,

$$\begin{aligned}
&\cos(\alpha+\beta) + i\sin(\alpha+\beta) \\
&\qquad = e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} \\
&\qquad = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\
&\qquad = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta),
\end{aligned}$$

and then equating real and imaginary parts finishes the proof.

Parity of sine and cosine

The function $\sin z$ is odd on all of $\mathbf{C}$:

$$\sin(-z) = \frac{e^{-iz} - e^{iz}}{2i} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z.$$

The function $\cos z$ is even on all of $\mathbf{C}$:

$$\cos(-z) = \frac{e^{-iz} + e^{iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

The Pythagorean identity, etc.

Another quick computation:

$$\cos^2 z + \sin^2 z = \cos(z - z) + i \sin(z - z) = e^{i(z-z)} = e^0 = 1.$$

From here, the algebra is identical to how one first learns the identities.

The circle constant

Define π *to be the smallest positive real number* θ *such that* $\sin(\theta) = 0$*.*

Consider

$$e^{i\pi} = \cos\pi + i\sin\pi.$$

By definition, $i\sin\pi = 0$. Furthermore, $|\cos\pi| = \sqrt{1 - \sin^2\pi} = 1$. But we also have

$$\cos(\pi) = 1 - 2\sin^2\left(\frac{\pi}{2}\right)$$

and by our definition $\sin(\frac{\pi}{2}) \neq 0$, hence $\sin^2(\frac{\pi}{2}) > 0$, hence $\cos(\pi) < 1$, hence $\cos(\pi) = -1$. This establishes

$$e^{i\pi} = -1,$$

commonly known as Euler's identity.

Perodicity

Define $\mathbf{e}(\theta) = e^{i\theta}$. Then

$$\sin(\theta) = \frac{\mathbf{e}(\theta) - \mathbf{e}(-\theta)}{2i}, \qquad \cos(\theta) = \frac{\mathbf{e}(\theta) + \mathbf{e}(-\theta)}{2}.$$

If $\mathbf{e}(\theta + p) = \mathbf{e}(\theta)$ for some period p, then $\mathbf{e}(-\theta + p) = \mathbf{e}(-\theta)$ and thus $\sin(\theta + p) = \sin(\theta)$ and $\cos(\theta + p) = \cos(\theta)$; that is, periodicity of $\mathbf{e}$ implies periodicity of sine and cosine.

Also, we have $\mathbf{e}(\alpha + \beta) = \mathbf{e}(\alpha)\mathbf{e}(\beta)$ as before.

We just established $\mathbf{e}(\pi) = -1$, hence $\mathbf{e}(2\pi) = \mathbf{e}(\pi)^2 = 1$, hence

$$\mathbf{e}(\theta + 2\pi) = \mathbf{e}(\theta)\mathbf{e}(2\pi) = \mathbf{e}(\theta).$$

That is: $\mathbf{e}$, sine, and cosine are all 2π–periodic.

Derivatives of trigonometric functions

We may compute the derivatives of sine and cosine:

$$\frac{d}{dz}[\sin z] = \frac{1}{2i}(ie^{iz} + ie^{-iz}) = \cos z,$$
$$\frac{d}{dz}[\cos z] = \frac{1}{2}(ie^{iz} - ie^{-iz}) = -\sin z.$$

Note that both cosine and sine satisfy the differential equation

$$f''(z) + f(z) = 0.$$

A claim involving $\pi/2$

We claim $\pi/2$ is the smallest positive real number θ such that $\cos(\theta) = 0$.

Suppose there were some positive $\theta' < \pi/2$ such that $\cos(\theta') = 0$.

Then $2\theta' < \pi$, and in particular

$$\sin(2\theta') = 2\sin(\theta')\cos(\theta') = 0,$$

contradicting our definition of π.

So our claim must be the case.

Halfway around the semicircle

To see $e^{i\pi/2} = i$, we note that $(e^{i\pi/2})^2 = -1$, hence $e^{i\pi/2} = \pm i$, which implies $\cos(\pi/2) = 0$. Furthermore,

$$|\sin(\pi/2)| = \sqrt{1 - \cos^2(\pi/2)} = 1.$$

It is actually kinda tricky to show that $\sin(\pi/2) = 1$:

Since $\cos(0) = 1$ and $\cos(\pi/2)$ is the smallest positive θ such that $\cos(\theta) = 0$, cosine is positive over $[0, \pi/2)$. Hence sine is increasing over $[0, \pi/2)$, and since $\sin(0) = 0$, sine is positive over $[0, \pi/2)$. If $\sin(\pi/2) < 0$ then by the intermediate value theorem, $\sin(\theta') = 0$ for some $\theta' < \pi$. That would be a contradiction: so we must have $\sin(\pi/2) > 0$. Since $|\sin(\pi/2)| = 1$, we must have $\sin(\pi/2) = 1$.

Remark

We just proved that $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$.

Since sine is odd, we get $\sin(-\pi/2) = -1$.

Since cosine is even, we get $\cos(-\pi/2) = 0$.

That is, $e^{-i\pi/2} = -i$.

What about smaller angles?

Essentially, this comes down to finding primitive roots of unity.

Computing $e^{i\pi/4}$

We now compute $e^{i\pi/4}$.

We know $(e^{i\pi/4})^2 = i$, so $e^{i\pi/4}$ is

$$\pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right),$$

and since cosine is positive over $[0, \pi/2)$, we have

$$e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

Hence, $\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

Computing $e^{i\pi/3}$

We now compute $e^{i\pi/3}$.

We know $(e^{\pi i/3})^3 = -1$, so $e^{\pi i/3} \in \{-1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \frac{1}{2} - \frac{\sqrt{3}}{2}i\}$.

Since sine is positive over $[0, \pi/2)$, we have

$$e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Hence, $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{\pi}{3}) = \frac{1}{2}$.

Computing $e^{i\pi/6}$

We now compute $e^{i\pi/6}$.

We want the complex number $a + bi$ on the unit circle such that

$$(a + bi)^2 = a^2 - b^2 + 2abi = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Now, $a^2 + b^2 = 1$ and $a^2 - b^2 = 1/2$ imply $2a^2 = 3/2$, hence $a = \frac{\sqrt{3}}{2}$, hence $b = \frac{1}{2}$. Putting things together, we get

$$e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

Hence, $\sin(\frac{\pi}{6}) = \frac{1}{2}$ and $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$.

Remark

One can continue in this way to find the values of trigonometric functions at all common angles.

Application: Dirichlet and Fejer kernels

Define the nth Dirichlet kernel as

$$D_n(x) := \sum_{k=-n}^{n} \mathbf{e}(kx).$$

Using the finite geometric sum formula,

$$\begin{aligned}
\sum_{k=-n}^{n} \mathbf{e}(kx) &= \sum_{k=0}^{n} \mathbf{e}(kx) + \sum_{k=0}^{n} \mathbf{e}(-kx) - 1 \\
&= \frac{\mathbf{e}((n+1)x) - 1}{\mathbf{e}(x) - 1} + \frac{\mathbf{e}(-(n+1)x) - 1}{\mathbf{e}(-x) - 1} - 1 \\
&= \frac{\mathbf{e}((n+1)x) - 1}{\mathbf{e}(x) - 1} + \frac{\mathbf{e}(x) - \mathbf{e}(-nx)}{\mathbf{e}(x) - 1} - \frac{\mathbf{e}(x) - 1}{\mathbf{e}(x) - 1} \\
&= \frac{\mathbf{e}((n+1)x) - \mathbf{e}(-nx)}{\mathbf{e}(x) - 1} \cdot \frac{\mathbf{e}(-\frac{x}{2})}{\mathbf{e}(-\frac{x}{2})} \\
&= \frac{\mathbf{e}((n+\frac{1}{2})x) - \mathbf{e}(-(n+\frac{1}{2})x)}{\mathbf{e}(\frac{x}{2}) - \mathbf{e}(-\frac{x}{2})} = \frac{\sin((n+\frac{1}{2})x)}{\sin(\frac{x}{2})}
\end{aligned}$$

Application: Dirichlet and Fejer kernels

The nth Fejer kernel is then

$$F_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(x).$$

We compute the following:

$$\begin{aligned}
F_n(x) &= \frac{1}{n}\sum_{k=0}^{n-1}\frac{\mathbf{e}((k+\frac{1}{2})x)-\mathbf{e}(-(k+\frac{1}{2})x)}{\mathbf{e}(\frac{x}{2})-\mathbf{e}(-\frac{x}{2})} = \frac{\mathbf{e}(\frac{x}{2})\sum_{k=0}^{n-1}\mathbf{e}(kx)-\mathbf{e}(-\frac{x}{2})\sum_{k=0}^{n-1}\mathbf{e}(-kx)}{n\left(\mathbf{e}(\frac{x}{2})-\mathbf{e}(-\frac{x}{2})\right)} \\
&= \frac{\frac{1}{\mathbf{e}(-\frac{x}{2})}\frac{\mathbf{e}(nx)-1}{\mathbf{e}(x)-1}-\frac{1}{\mathbf{e}(\frac{x}{2})}\frac{\mathbf{e}(-nx)-1}{\mathbf{e}(-x)-1}}{n\left(\mathbf{e}(\frac{x}{2})-\mathbf{e}(-\frac{x}{2})\right)} = \frac{\frac{\mathbf{e}(nx)-1}{\mathbf{e}(\frac{x}{2})-\mathbf{e}(-\frac{x}{2})}+\frac{\mathbf{e}(-nx)-1}{\mathbf{e}(\frac{x}{2})-\mathbf{e}(-\frac{x}{2})}}{n\left(\mathbf{e}(\frac{x}{2})-\mathbf{e}(-\frac{x}{2})\right)} \\
&= \frac{\mathbf{e}(nx)-2+\mathbf{e}(-nx)}{n\left(\mathbf{e}(\frac{x}{2})-\mathbf{e}(-\frac{x}{2})\right)^2} = \frac{\left(\mathbf{e}(\frac{nx}{2})-\mathbf{e}(-\frac{nx}{2})\right)^2}{n\left(\mathbf{e}(\frac{x}{2})-\mathbf{e}(-\frac{x}{2})\right)^2} = \frac{\sin^2(\frac{nx}{2})}{n\sin^2(\frac{x}{2})}.
\end{aligned}$$