Mathematical Spaces

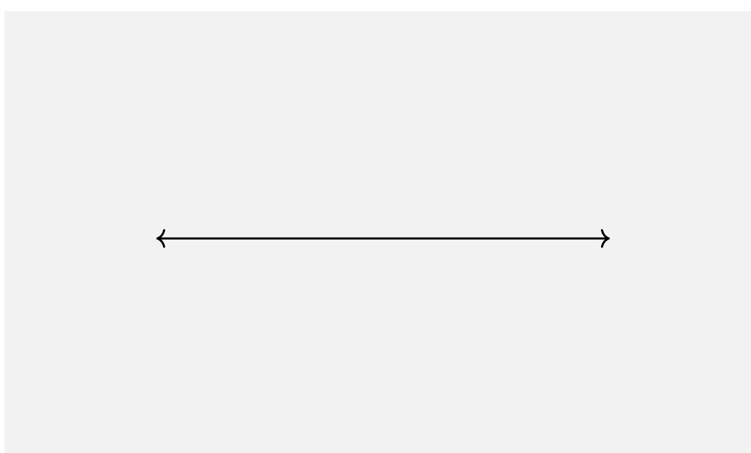
Mathematics is written in terms of sets, but imagined in terms of spaces. There is no definition of space; rather, the word is a (heavily used) informal term.

Euclidean space

The simplest space is the point:[1]

The point has no width, height, or length. Informally, it has no dimension: it merely denotes a location. One can think of the point as **Euclidean 0-space**, though no one calls it that. The set associated to this space is often denoted $\{0\}$.

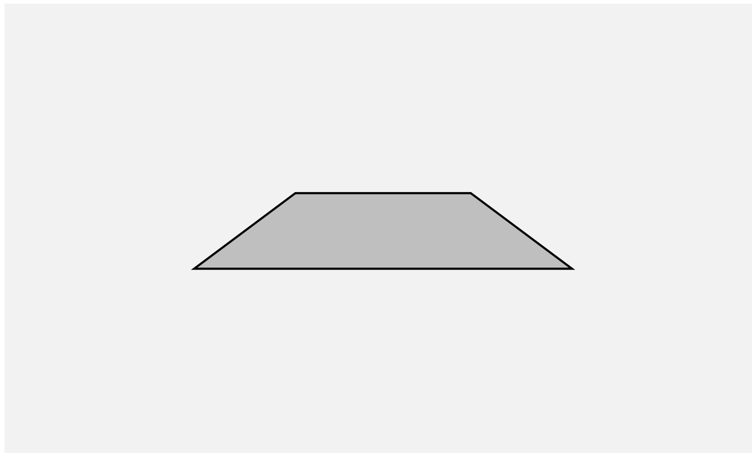
The next simplest space is the line:



The line has no width, no height, and infinite length. Two distinct points determine a unique line, and any line contains an infinite number of points. In fact, there are so many points on the line that it is impossible to put them all in a sequence. One can think of the line as **Euclidean 1-space**, though no one calls it that. The set associated to this space is often denoted $\mathbf{R}$, and elements of this set are called "real numbers."

[1] I'm going to draw spaces against a very light grey background so that they are easier to distinguish.

The next simplest space is the plane:



The plane has no height, but infinite length and width. A line ℓ and a point P not on ℓ determine a unique plane, and any plane contains an infinite number of lines and points. This is **Euclidean 2-space**, and indeed some people do call it that. The set associated to this space is often denoted $\mathbf{C}$, and elements of this set are called "complex numbers."

We can keep adding dimensions, though the spaces get harder and harder to draw.

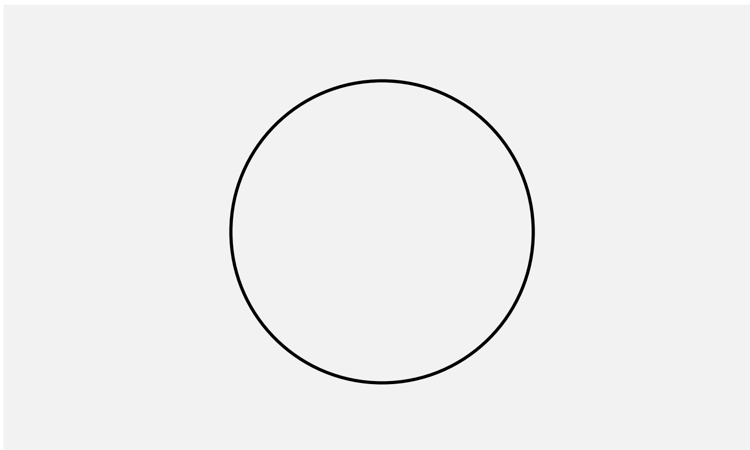
We also have the void:

This could be thought of as **Euclidean (-1)-space** (and almost certainly no one calls it that). There is not even a location defined: this space carries no information whatsoever. The set associated to this space is often denoted $\varnothing$.

Compactification

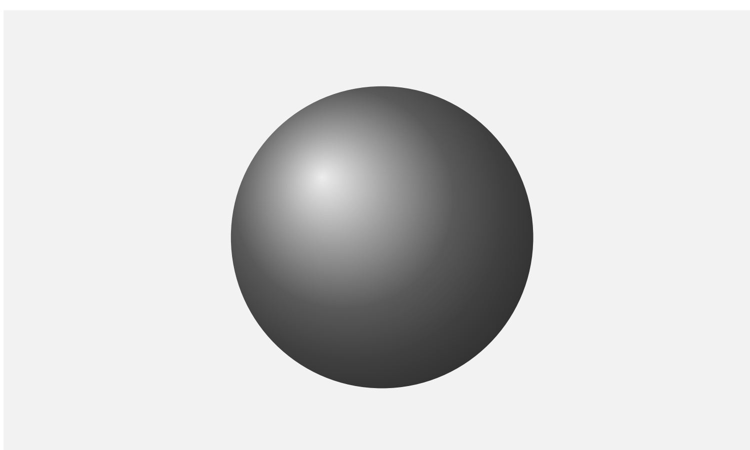
The line and the plane are very useful spaces. However, they spawl, which can be inconvenient. Sometimes it is ideal to work in a space that does not sprawl. This condition of not-sprawling (combined with some other technical hypotheses) is called being **compact**.

We can make a line compact by giving it curvature. If a point moves back and forth through space along a path with zero curvature, the resulting trail forms a line. If a point moves back and forth through space along a path with positive constant curvature, the resulting trail forms a circle.



This space is **non-Euclidean**, since it bends. The set associated to this space is often denoted S^1.

The set S^1 can be defined as the set of points a unit radius away from an origin in Euclidean 2-space. Similarly, we may define the set of points a unit radius away from an origin in Euclidean 3-space.



This forms a sphere (which is also non-Euclidean), sometimes called the **2-sphere** to distinguish it from other similarly defined objects. The set associated to this space is often denoted S^2.

Just as before: we can carry this construction out in higher dimensions, though the spaces get harder and harder to draw.