An Easy Instance of Simultaneous Diagonalization

Let ρ be a planar rotation of $\pi/3$ radians, and σ a planar rotation of $\pi/4$ radians:

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}, \qquad \sigma = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

We will use these to construct two commuting nondegenerate operators.

Let $\Omega = \rho \oplus \mathbf{1}$ be rotation of the xy-plane, and $\Lambda = \mathbf{1} \oplus \sigma$ a rotation of the zw-plane.

Then

$$[\Omega, \Lambda] = (\rho \oplus \mathbf{1})(\mathbf{1} \oplus \sigma) - (\mathbf{1} \oplus \sigma)(\rho \oplus \mathbf{1}) = \rho \oplus \sigma - \rho \oplus \sigma = \mathbf{0}.$$

We have

$$\det(\rho - r\mathbf{1}) = \left(\frac{1}{2} - r\right)^2 + \frac{3}{4} = r^2 - r + 1 = (r - e^{i\pi/3})(r - e^{-i\pi/3})$$

and

$$\det(\sigma - s\mathbf{1}) = \left(\frac{1}{\sqrt{2}} - s\right)^2 + \frac{1}{2} = s^2 - \sqrt{2}s + 1 = (s - e^{i\pi/4})(s - e^{i\pi/4})$$

so all together, we have $\omega = 1, 1, e^{i\pi/3}, e^{-i\pi/3}$ and $\lambda = 1, 1, e^{i\pi/4}, e^{-i\pi/4}$.

Just as in previous HW, ρ and σ have the same eigenvectors.

One can check that

$$|\omega = e^{i\pi/3}\rangle = |\lambda = e^{i\pi/4}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

and similarly

$$|\omega = e^{-i\pi/3}\rangle = |\lambda = e^{-i\pi/4}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Now let $\Psi = \frac{1}{\sqrt{2}}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, a unitary matrix with inverse $\Psi^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$.

Finally, let $U = \Psi \oplus \Psi$, so that $U^{-1} = \Psi^{-1} \oplus \Psi^{-1}$.

$$U^{-1}\Omega U = (\Psi^{-1} \oplus \Psi^{-1})(\rho \oplus \mathbf{1})(\Psi \oplus \Psi) = (\Psi^{-1}\rho\Psi) \oplus \mathbf{1}$$

$$U^{-1}\Lambda U = (\Psi^{-1} \oplus \Psi^{-1})(\mathbf{1} \oplus \sigma)(\Psi \oplus \Psi) = \mathbf{1} \oplus (\Psi^{-1}\sigma\Psi)$$

But

$$\Psi^{-1}\rho\Psi = \begin{bmatrix} e^{i\pi/3} & 0 \\ 0 & e^{-i\pi/3} \end{bmatrix}, \qquad \Psi^{-1}\sigma\Psi = \begin{bmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix}$$

so $U^{-1}\Omega U$ and $U^{-1}\Lambda U$ are clearly diagonal.