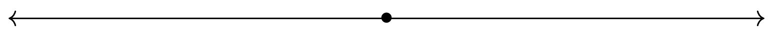
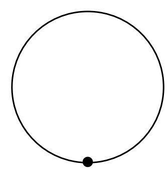
Introduction

Investigating the behavior of the circle is just as interesting as investigating the behavior of the line.

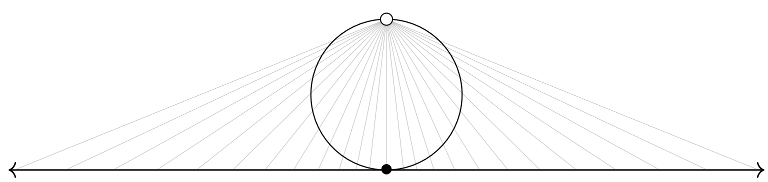
There are exactly two sorts of continuous one-dimensional shapes which are compatible with some sort of algebraic structure. The first shape is a line with a basepoint (as explored in Arithmetic of the Continuum):



The second shape is a circle with a basepoint:



We can pair points on the circle with points on the line in near-exact correspondence, so that the basepoints coincide, with only one circle point having no corresponding line point:



From a topological perspective these are different objects: a line has no holes in it, whereas a circle can be thought of as consisting of a hole and its boundary. Further, from a metric perspective, every line in the plane is unbounded, whereas every circle in the plane is bounded by its radius.

Yet the two are directly related from a more algebraic perspective: the circle is *birationally equivalent* to the line, i.e. we have the maps between the line and circle given by

$$\mathtt{LINE}(x, y) = \frac{1-y}{x}, \qquad \mathtt{CIRCLE}(t) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$$

where t describes points on the line and (x, y) with $x^2 + y^2 = 1$ describes points on the circle. This is more or less what's happening when we remove a point from the circle and then unravel/extend the resulting arc until it forms a line.

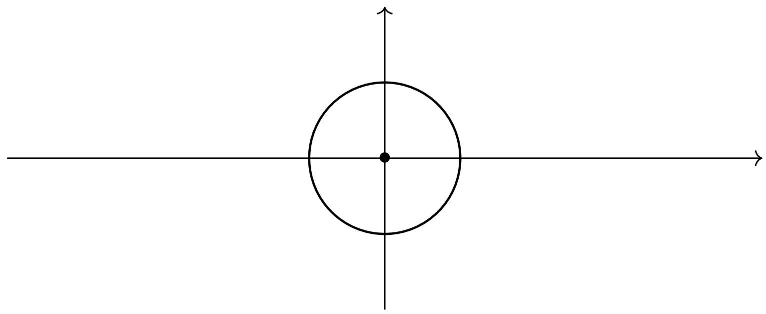
One can think of the circle and the line as **complementary structures:** whereas smooth images of the line are useful for modeling *trajectories*, smooth images of the circle are useful for modeling *oscillations.*

Together, the circle and the line form a comprehensive description of continuous one-dimensional motion.

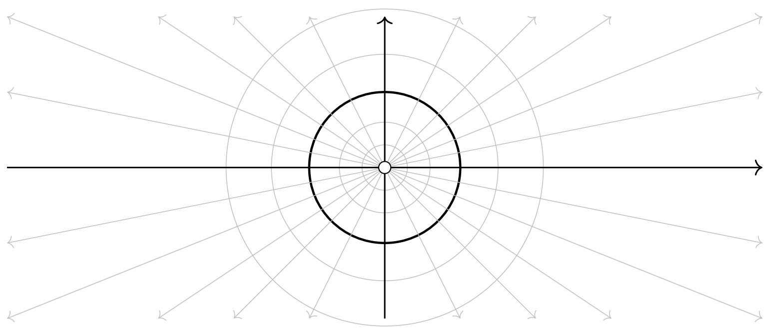
Coordinates in the Plane

A very natural circle to study in the Euclidean plane $(\mathbf{R}^2, \langle\cdot,\cdot\rangle_{\mathsf{dot}})$ is the boundary of the unit disc:

$$\partial\mathbf{D} := \partial B(0,1) = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$$



We call this circle the **unit circle.** If we delete the origin of the plane, every remaining point on the plane can be uniquely described by specifying a point on the unit circle and a positive factor by which to scale the unit circle:



Thus, the change of coordinates

$$\phi : \mathbf{R}^2 \setminus \{(0,0)\} \to (0,\infty) \times \partial\mathbf{D} \qquad \text{via} \qquad (x,y) \mapsto \Big(\|(x,y)\|,\ \mathtt{atan2}(y,x)\Big)$$

is well-defined for all $(x,y) \in \mathbf{R}^2 \setminus \{(0,0)\}$. The well-defined inverse change of coordinates is then

$$\psi : (0,\infty) \times \partial\mathbf{D} \to \mathbf{R}^2 \setminus \{(0,0)\} \qquad \text{via} \qquad (r,\theta) \mapsto (r\cos\theta,\ r\sin\theta).$$

Now, define for two points (a,b) and (c,d) in $\mathbf{R}^2$ (including $(0,0)$, the plane origin) the multiplication

$$(a,b)\cdot(c,d) = (ac - bd, ad + bc)$$

so that multiplying any point by $(0,1)$ corresponds to rotating that point in the plane by a quarter-turn in the counterclockwise direction, an operation compatible with $\langle\cdot,\cdot\rangle_{\mathsf{dot}}$. Every point besides $(0,0)$ is then invertible.

The resulting plane (with the origin included), paired with such a multiplication, forms a structure called the *complex plane*, denoted $\mathbf{C}$. The study of $\mathbf{C}$–valued functions defined on $\mathbf{C}$ is known as **complex analysis**.

Groups and G–Spaces

Complex analysis, roughly, is to real analysis as algebra is to analysis itself. That is, it is a more rigid subject than its counterpart, and this rigidity leads to deep structure. This is why we begin our journey through complex analysis with the definition of a group (normally introduced in an algebra course).

Definition 0.1. A **group** is a triple $(G, \circ, e)$ wherein: G is a set, $\circ : G^2 \to G$ is a law of composition, and e is a special element of G, neutral with respect to $\circ$. These three objects satisfy the following properties:

1. The composition law $\circ$ is associative, i.e. does not care about the grouping of the terms, so that

   $$(g \circ g') \circ g'' = g \circ (g' \circ g'')$$

   for all $g, g', g'' \in G$. Thus, we may write the above unambiguously as $g \circ g' \circ g''$.

2. The identity is indeed $\circ$-neutral:

   $$g \circ e = g = e \circ g$$

   for all $g \in G$.

3. Each element $g \in G$ has a corresponding opposite/inverse element $g^{-1} \in G$ with respect to $\circ$:

   $$g \circ g^{-1} = e = g^{-1} \circ g$$

Groups on their own are lofty abstract entities, so we tether them to reality by letting them act on ordinary sets.

Definition 0.2. Let G be a group, X a nonempty set. We say G **acts** on X (or that X is a **G–space**) if there exists

$$\alpha : G \times X \to X,$$

called the *action*, satisfying the following properties:

- $\alpha(e, x) = x$
- $\alpha(g, \alpha(h, x)) = \alpha(g \circ h, x)$.

An equivalent viewpoint is that an action consists of a family of maps

$$\tau_g : X \to X,$$

one for each element $g \in G$, such that:

- $\tau_e(x) = x$
- $\tau_g(\tau_h(x)) = \tau_{g \circ h}(x)$.

Thus, "abstract groups" form the trivial case where $G = X$ and the action is simply $\alpha(g, g') = g \circ g'$.

Faithful, Free, Transitive

A group action is said to be:

- **Faithful** if $\tau_g(x) = x$ for all $g \in G$ implies $g = e$,
- **Free** if $\tau_g(x) = x$ for some $g \in G$ implies $g = e$,
- **Transitive** if for any two points $x, y \in X$ there is some $g \in G$ such that $\tau_g(x) = y$.

If G acts transitively on X, we say that X is a **homogeneous G–space** (picture a space with some sort of fluid flowing through it). If G acts transitively and freely on X, we say that X is a *principal homogeneous* G*–space* or, more succinctly, a **G–torsor** (here, the fluid is can be thought of as incompressible).

The difference between a G–torsor and the group G itself is that G has an distinguished basepoint, whereas a G–torsor does not. Thus, it makes sense to speak of differences or ratios within a G–torsor, but not sums or products, since the latter operations assume a neutral element.

Examples of Groups and G–Spaces

- The simplest example of a group is a point:

  $$G = \{e\},$$

  often called the *trivial group*.

- The line $(\mathbf{R}, +, 0)$ and its positive elements $(\mathbf{R}^+, \cdot, 1)$ both form groups. In fact, using the maps

  $$x \mapsto \sum_{n \geq 0} \frac{x^n}{n!} = \exp x, \qquad y \mapsto \int_1^y \frac{dt}{t} = \log y$$

  it can be seen that $(\mathbf{R}, +, 0)$ and $(\mathbf{R}^+, \cdot, 1)$ form essentially the same group: $\mathbf{R}$ is additive, whereas $\mathbf{R}^+$ is multiplicative, and the exponential function is the map that converts addition into multiplication. This is likely why Rudin called the exponential map "the most important function in mathematics."

- Extending this, any vector space V over $\mathbf{R}$ has an additive group structure built into it. The multiplicative group structure on a subset of V and a corresponding exponential map aren't as easy to generalize, except in very special cases (see below).

- Given a vector space V, let $G = GL(V)$ (the group of invertible linear maps on V) and let X be the set of all possible bases for V. This forms a torsor, since without further structure on V, no basis can claim to be the one true basis for V.

- Indefinite integration in one variable operates on an $\mathbf{R}$–torsor: given an antiderivative $\int f$ of f, one can add a constant C to it to obtain another antiderivative $\int f + C$, yet no antiderivative can truly claim to be *the* antiderivative of f.

- Voltage differences are elements of $\mathbf{R}$, but voltages themselves are elements of an $\mathbf{R}$–torsor.

$U(1)$

The circle is the group obtained by setting $2\pi = 0$ in the group $(\mathbf{R}, +, 0)$, thus forming the quotient $\frac{\mathbf{R}}{2\pi\mathbf{Z}}$.

We will denote this group by $U(1)$. Picture a circle. We define:

$$\mathbf{e}(\theta) = \sum_{n \geq 0} \frac{(2\pi i\theta)^n}{n!}, \qquad \overline{\mathbf{e}(\theta)} = \mathbf{e}(-\theta).$$

where i is a nonreal number such that $i^2 = -1$. (Note that we've casually extended the definition of the exponential map to include not only all of $\mathbf{R}$ but every point of the form $x + iy$ where $x, y \in \mathbf{R}$. This requires justification, but for now let's focus on the geometry.)

The map $\mathbf{e}(\theta)$ represents rotating by θ counterclockwise, whereas $\overline{\mathbf{e}(\theta)}$ represents rotating by θ clockwise.

As a consequence of the binomial theorem,

$$\mathbf{e}(\alpha + \beta) = \mathbf{e}(\alpha)\mathbf{e}(\beta)$$

and hence

$$\mathbf{e}(N\theta) = \mathbf{e}(\theta)^N.$$

Also note that

$$1 = \mathbf{e}(0) = \mathbf{e}(\theta - \theta) = \mathbf{e}(\theta)\mathbf{e}(-\theta)$$

so that

$$\mathbf{e}(-\theta) = \mathbf{e}(\theta)^{-1}, \qquad \|\mathbf{e}(\theta)\| = 1.$$

The circle can be thought of in an abstract sense, but usually people mean to embed it within the plane $\mathbf{R}^2$, identifying it with all points $(x, y) \in \mathbf{R}^2$ such that $x^2 + y^2 = 1$. Such points can be described in terms of an angle θ by setting $x = \cos(2\pi\theta)$ and $y = \sin(2\pi\theta)$. The resulting object is called the **unit circle**. Thus we have:

$$\mathbf{e}(\theta) = \cos(2\pi\theta) + i\sin(2\pi\theta), \qquad \overline{\mathbf{e}(\theta)} = \cos(2\pi\theta) - i\sin(2\pi\theta)$$

so that

$$\cos(2\pi\theta) = \frac{\mathbf{e}(\theta) + \overline{\mathbf{e}(\theta)}}{2}, \qquad \sin(2\pi\theta) = \frac{\mathbf{e}(\theta) - \overline{\mathbf{e}(\theta)}}{2i}.$$

Roots of Unity

These are the points obtained from cutting the unit circle into evenly spaced sectors, where one of the cuts is at 1.

Algebraically, a **primitive Nth root of unity** is a solution ζ to $z^N = 1$ where $\zeta^M \neq 1$ for any positive $M < N$. Thus, we may find certain primitive roots of unity by factoring $z^N - 1$. Similarly, the inability to write certain primitive roots of unity in terms of radicals is exactly the case of which $z^N - 1$ factor in terms of radicals.

Fortunately, the most frequently encountered roots of unity are of low degree. Here they are.

- Primitive first, second, and fourth roots of unity:

$$\mathbf{e}(0) = 1, \qquad \mathbf{e}(1/4) = i, \qquad \mathbf{e}(1/2) = -1, \qquad \mathbf{e}(3/4) = -i.$$

- Primitive third and sixth roots of unity:

$$\mathbf{e}(1/6) = \frac{1 + i\sqrt{3}}{2}, \qquad \mathbf{e}(1/3) = \frac{-1 + i\sqrt{3}}{2}, \qquad \mathbf{e}(2/3) = \frac{-1 - i\sqrt{3}}{2}, \qquad \mathbf{e}(5/6) = \frac{1 - i\sqrt{3}}{2}.$$

- Primitive fifth roots of unity:

$$\mathbf{e}(1/5) = \frac{(\sqrt{5} - 1) + i\sqrt{10 + 2\sqrt{5}}}{4}, \qquad \mathbf{e}(2/5) = \frac{-(\sqrt{5} + 1) + i\sqrt{10 - 2\sqrt{5}}}{4},$$

$$\mathbf{e}(3/5) = \frac{-(\sqrt{5} + 1) - i\sqrt{10 - 2\sqrt{5}}}{4}, \qquad \mathbf{e}(4/5) = \frac{(\sqrt{5} - 1) - i\sqrt{10 + 2\sqrt{5}}}{4}.$$

- Primitive eighth roots of unity:

$$\mathbf{e}(1/8) = \frac{1 + i}{\sqrt{2}}, \qquad \mathbf{e}(3/8) = \frac{-1 + i}{\sqrt{2}}, \qquad \mathbf{e}(5/8) = \frac{-1 - i}{\sqrt{2}}, \qquad \mathbf{e}(7/8) = \frac{1 - i}{\sqrt{2}}$$

So, for concreteness, i is exactly where 1 ends up after a quarter-turn counterclockwise, and -1 is exactly where i ends up after a quarter-turn counterclockwise, since

$$-1 = \mathbf{e}(1/2) = \mathbf{e}(1/4 + 1/4) = \mathbf{e}(1/4)\mathbf{e}(1/4) = i \cdot i.$$

This is what it looks like for $U(1)$ to act on itself as a $U(1)$–space: the left $\mathbf{e}(1/4)$ lives in G, whereas the right $\mathbf{e}(1/4)$ lives in X, and it just so happens that in this particular case we have $G = X$.

Moreover, each collection of nth roots of unity is the isomorphic to a cyclic (i.e. generated by a single element) group of order n, denoted $(\mathbf{Z}/n\mathbf{Z}, +)$. That is, the nth roots of unity form a complex representation of $\mathbf{Z}/n\mathbf{Z}$.