The Plane

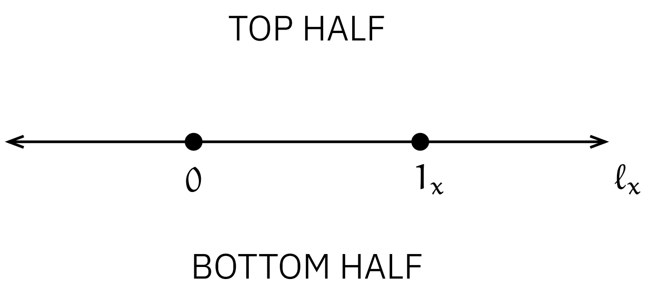
an introduction to coordinates

Picture a sheet of paper, so large that a circle of any size drawn in pen can fit on the sheet. Now, imagine placing two pins on the sheet.

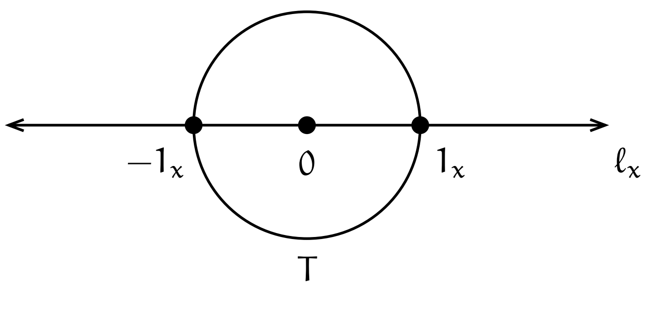
Call the first pin 0 and the second pin 1_x.

Use a ruler to draw a straight line segment in pen that connects the dots 0 and 1_x. Exactly one line exists that extends the segment just drawn while bisecting the sheet into two equal halves. Orient your perspective so that on this line, 0 is left of 1_x and 1_x is right of 0.

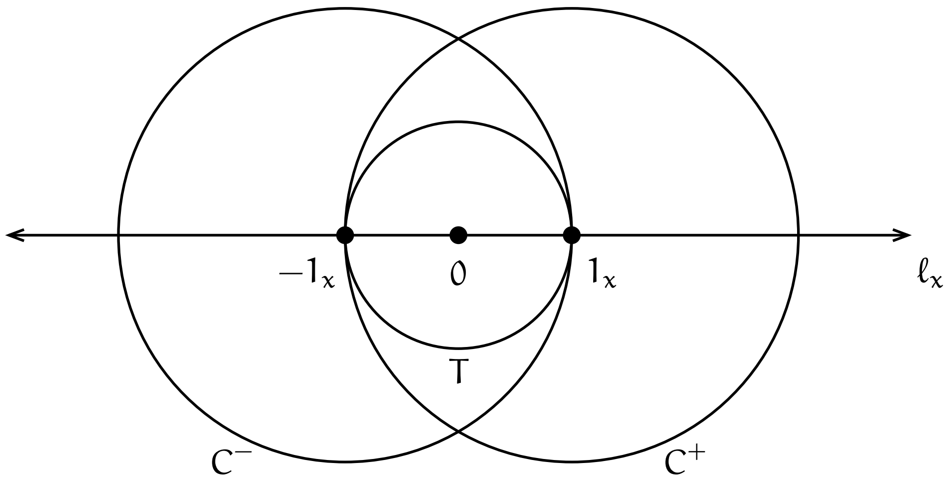
We will call the half above the line the top half, the half below the line the bottom half, and the line itself ℓ_x.



Now, draw a circle whose center coincides with 0 and whose edge contains 1_x. Call the circle T. The edge of T also contains another point on the line ℓ_x besides 1_x. Call this point, which is left of 0, -1_x.

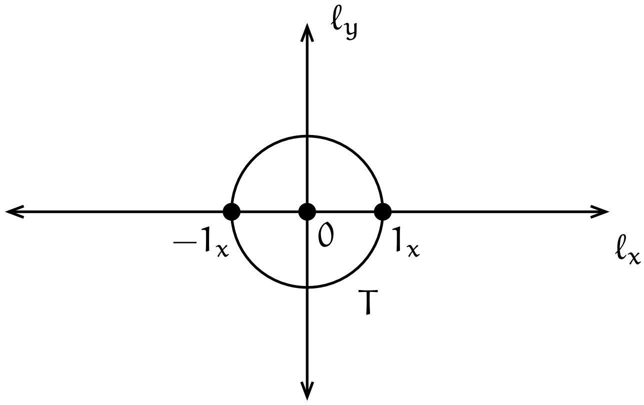


We will now draw two larger circles. Let the first circle, which we'll call C^{-}, have center coinciding with -1_x, with edge containing the point 1_x. Let the second circle, which we'll call C^{+}, have center coinciding with 1_x, with edge containing the point -1_x. The circle T should lie within the overlap of C^{-} and C^{+}, tangent to C^{-} at the point 1_x and tangent to C^{+} at the point -1_x.

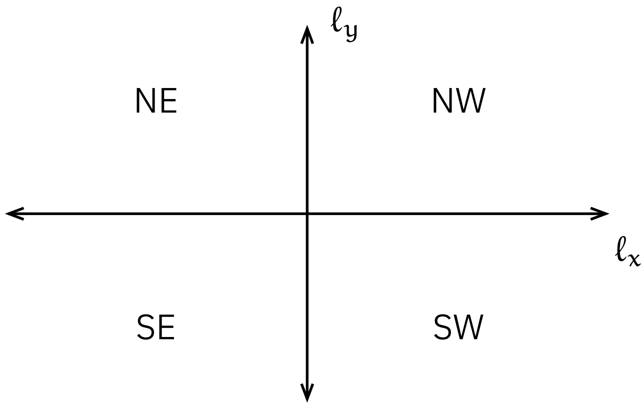


The circles C^{-} and C^{+} intersect (meet) at two points on the sheet. Neither of these points lies on the line ℓ_x. Use a ruler to draw a straight line segment containing both of these points. There is a unique line extending this line segment, call it ℓ_y.

The line ℓ_y intersects with T at two points: one point on the top half of the sheet, one point on the bottom half of the sheet. Call the point on the top half 1_y and the point on the bottom half -1_y. Now we can erase C^{-} and C^{+}, as we no longer need them.



The lines ℓ_x and ℓ_y meet at exactly one point: 0. The lines ℓ_x and ℓ_y also divide the sheet into four equal quadrants: NE, NW, SW, and SE.

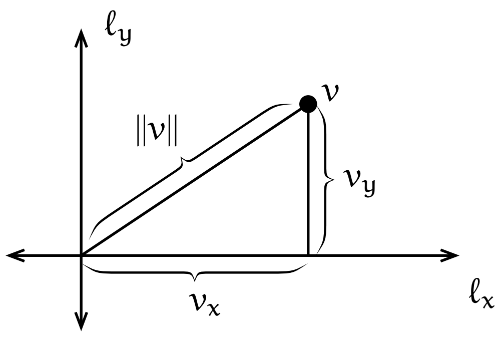


Before we continue: some of the constructions above have standard names.

- The sheet of paper we begun with is often called the *plane.*
- The point 0 is often called the *origin.*
- The point 1_x is often called $(1, 0)$; 1_y is often called $(0, 1)$.
- The line ℓ_x is often called the x-axis; ℓ_y is often called the y-axis.
- The circle T is often called the unit circle.

Any point on the inside of any plane quadrant can be described by a unique right triangle whose base leg is contained within the x-axis and whose height leg runs parallel to the y-axis. Suppose we have such a point; call the point v.

Draw the corresponding right triangle for v. We denote the length of the base leg of this triangle by v_x and the length of the height leg of this triangle by v_y. The length of the hypotenuse of this triangle is denoted $\|v\|$. We note that v_x is often called the x-coordinate of v and that v_y is often called the y-coordinate of v.



The number $\|v\|$ is often called the norm of v. Those who are familiar with the Pythagorean theorem may notice that

$$\|v\|^2 = v_x^2 + v_y^2.$$

Questions to think about:

1. When $\|v\| = 1$, where is v in the plane? What about when $\|v\| < 1$? What about when $\|v\| > 1$?
2. What value ought v_x take when v is not inside of a quadrant but instead on the y-axis? What value ought v_y take when v is not inside of a quadrant but instead on the x-axis? What value should v_x and v_y equal when $v = 0$?
3. Why is the height leg automatically parallel to the y-axis? How did we accomplish this?
4. Why is the right triangle corresponding to each point on the inside of a plane quadrant unique?

Written 20 Aug 2026 by Justin T Chun.
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