Geometry of Projective Space

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> Until and unless some totally new principle is discovered, the subject of synthetic projective geometry is no longer a very fruitful field for original research. On the other hand, it would be a disaster to the whole geometric fabric if a time ever came when synthetic methods were completely abandoned. Not only do they have a permanent beauty which no one who has ever studied them can forget, but they afford an invaluable insight into the inner significance of geometrical science and an invaluable training for any geometer. If Plato wrote over the gate of the Academy, "Let none ignorant of geometry presume to enter here," surely we may write to-day in the same spirit, "Let none ignorant of the fundamentals of synthetic projective geometry presume the title of geometer."
>
> – J. L. Coolidge, *The Rise and Fall of Projective Geometry*

In these notes we sketch an outline of projective geometry (henceforth simply "geometry"). Though the quotation above references synthetic geometry, we will also explore coordinate geometry, with the goal thorough understanding of the content referenced in the first chapter of [BCG09].

Our main reference is [Ben95].

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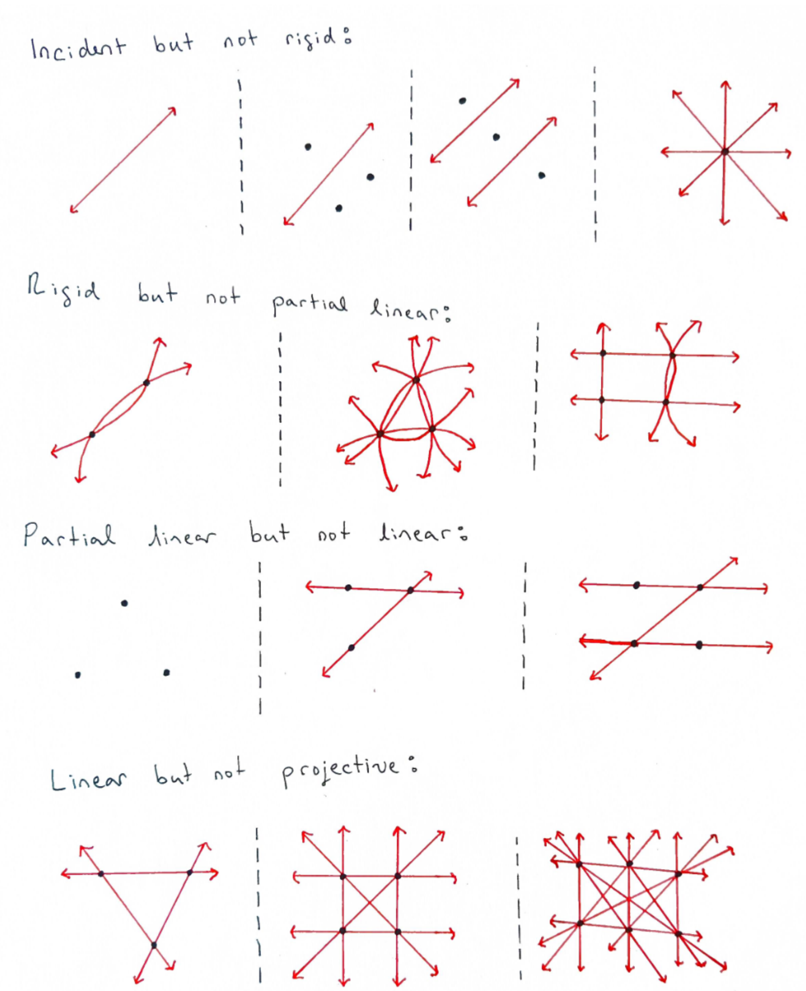
Part I
Synthetic Geometry



1 Points and lines

We start by describing the bare minimum required to reason geometrically.

Defintion 1.1: Incidence structure, (partial) linear space

An **incidence structure** is a triple $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ where $\mathcal{P}$ and $\mathcal{L}$ are sets and $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is a binary relation. Elements of $\mathcal{P}$ are called *points*, elements of $\mathcal{L}$ are called *lines*, and elements of I are called *flags*.

- An incidence structure S is a **rigid structure** if any line is incident with at least two points.
- A rigid structure S is a **partial linear space** if any pair of points is incident with at most one line.
- A partial linear space S is a **linear space** if any pair of points is incident with at least one line.

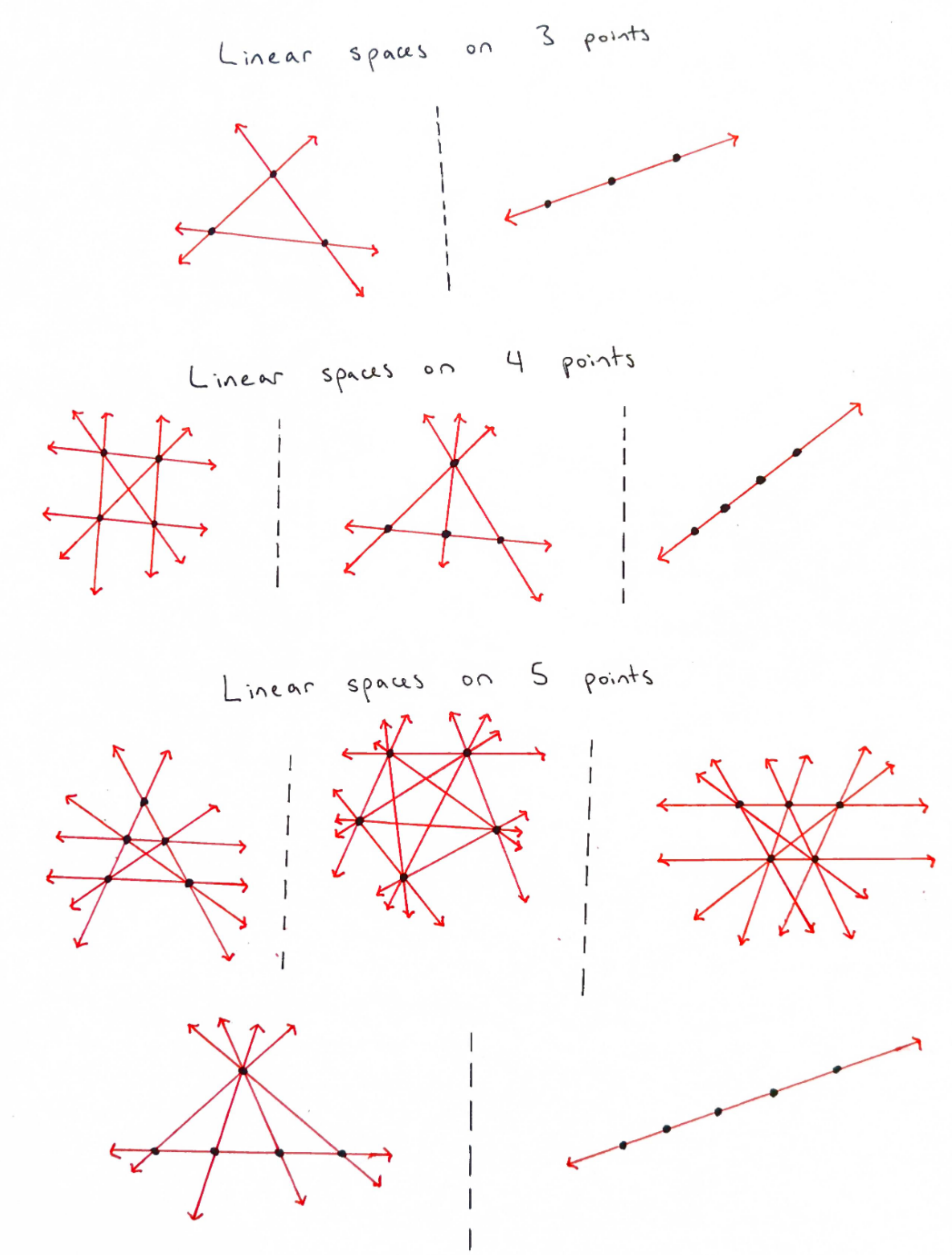
A linear space S is *nontrivial* if $|\mathcal{L}| > 1$. For $P \in \mathcal{P}$ and $\ell \in \mathcal{L}$, we define $P \leq \ell$ to mean $(P, \ell) \in \mathrm{I}$.

Note that in a linear space S,

every $P_0, P_1 \in \mathcal{P}$ *where* $P_0 \neq P_1$ *has a unique line* $\ell \in \mathcal{L}$ *satisfying both* $P_0 \leq \ell$ *and* $P_1 \leq \ell$*.*

We denote this unique line $P_0 \vee P_1$.

When each line is incident with exactly two distinct points, we get a complete graph. In fact, every linear space with n points can be formed by flattening K_n in a certain way.



2 Subspaces of linear spaces, span of a subset

Given a set of points $\mathcal{P}$, we can construct the lines $\mathcal{L}$ of a linear space S by pairing all the points in $\mathcal{P}$ with a line. However, this does not uniquely determine a linear space: we also need to specify the incidence relation I, which can be thought of as a set of instructions for flattening K_n.

This gives us a clue as to how to define subspaces:

a subspace ought to inherit the incidence of its corresponding superspace.

Defintion 2.1

Let $S = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a linear space. A **subspace** is a set $S' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ where $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}'$ is induced from $\mathcal{P}'$, and I' is I restricted to $\mathcal{P}'$. We write $S' \leq S$.

This ought to also be a linear space unless something goes horribly wrong. Fortunately, all is well:

Theorem 2.2

Subspaces of a linear space are linear spaces.

Proof. Suppose $S' \leq S$ for some linear space S. We need to check that every line in $\mathcal{L}'$ is incident with at least two points in $\mathcal{P}'$, and that every pair of distinct points in $\mathcal{P}'$ have exactly one line in $\mathcal{L}'$ going through them. But these both follow by $\mathcal{L}'$ was induced from $\mathcal{P}'$. $\square$

Similar comment as above, but for intersections of subspaces:

Theorem 2.3

Subspaces of a linear space are stable under arbitrary intersection.

Proof. This is basically an exercise in figuring out what "arbitrary intersection" ought to mean. Suppose J is some index set, and $\{S_\iota\}_{\iota \in J}$ a collection of subspaces, indexed by J, of some linear space S. Then let S_J be the subset of S whose points are

$$\mathcal{P}_J = \bigcap_{\iota \in J} \mathcal{P}_\iota,$$

whose lines $\mathcal{L}_J$ are induced from $\mathcal{P}_J$, and whose incidence relation is I restricted to $\mathcal{P}_J$. This is linear. $\square$

Defintion 2.4

For any subset $S'' \subseteq S$ we may define $\langle S'' \rangle$, called the **span** of S'', to be

$$\langle S'' \rangle = \bigcap_{S'' \subseteq S' \leq S} S'.$$

3 Linear spaces: bases and dimension

We define bases as minimal spanning sets.

Defintion 3.1

Let S be a linear space, $S' \leq S$.

A **basis** of S' is a set of points E such that $\langle E \rangle = S'$ and $\langle E \setminus e \rangle \subsetneq S'$ for all $e \in E$.

The **dimension** of S' is $r_{S'} = \max\{|E| - 1 : E \text{ is a basis of } S'\}$.

We may shorten this to just r when the space is evident.

Here is a table with the names of what we shall call linear spaces of dimension r.

r	name
-1	void
0	point
1	line
2	plane
3	solid

4 Projective planes

Defintion 4.1

A linear space S is a **projective space** if:

- Every line has at least three points on it.
- Vemblen's axiom: Given distinct points $A, B, C, D \in \mathcal{P}$, if there is some $E \in \mathcal{P}$ such that $E \leq A \vee B$ and $E \leq C \vee D$, then we have both $F \leq A \vee C$ and $F \leq B \vee D$ for some $F \in \mathcal{P}$.

If, further,

- There are four points, no three of which are collinear.

then S is *nondegenerate* (i.e. is at least a plane).

A **projective plane** is a nondegenerate projective space of dimension 2.

4.1 Plane duality

Theorem 4.2

In a projective plane S, any two distinct lines ℓ_0 and ℓ_1 intersect in exactly one point (denoted $\ell_0 \wedge \ell_1$).

Proof. If ℓ_0 and ℓ_1 did not intersect at all, then we could take two points on ℓ_0 and two points on ℓ_1 and get a basis for a subspace of S, implying S has dimension at least 3. But this contradicts the assumption that S is a projective plane, so ℓ_0 and ℓ_1 intersect at some point. This point must be unique: were there two points on both lines, then the lines would coincide (uniqueness is forced by the existence of the two points), contradicting the assumption that the lines were distinct. □

4.2 Pencils and ranges

Defintion 4.3

For a point P, denote by $*P$ the set of all lines ℓ such that $P \leq \ell$. Such a set is called a **pencil** of lines.

For a line ℓ, denote by $*\ell$ the set of all points P such that $P \leq \ell$. Such a set is called a **range** of points.

I'm borrowing this notation more or less from Python syntax.

4.3 Triangles, complete quadrilaterals, complete quadrangles

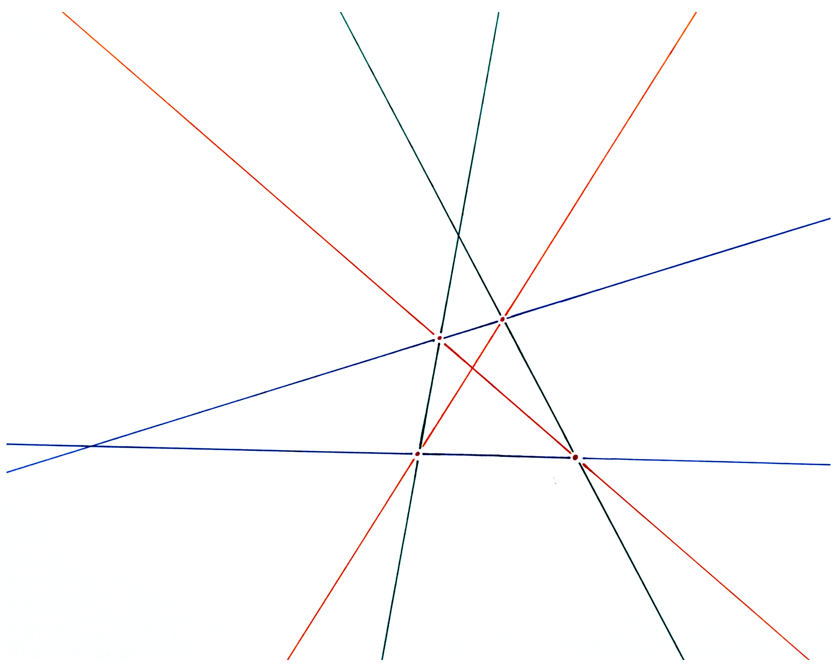
Defintion 4.4

Let J be some index set and $\mathcal{S}$ a projective plane.

- Points $\{P_\iota\} \subseteq \mathcal{P}$ are **collinear** if there is some $\ell \in \mathcal{L}$ for which $P_\iota \leq \ell$ for all $\iota \in J$.
- Lines $\{\ell_\iota\} \subseteq \mathcal{L}$ are **copunctual** if there is some $P \in \mathcal{P}$ for which $P \leq \ell_\iota$ for all $\iota \in J$.
- A **triangle** consists of three noncollinear points (or equivalently three noncopunctual lines).
- A **complete quadrilateral** consists of four lines meeting in six distinct points.
- A **complete quadrangle** consists of four points joined by six distinct lines.

4.4 A closer look at Vemblen's axiom

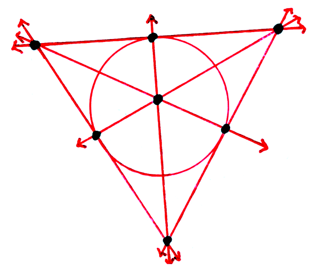
This says for any four points A, B, C, D that $A \vee B$ and $C \vee D$ intersecting implies $A \vee C$ and $B \vee D$ intersecting. Applying this to A, C, D, B, we see that $A \vee C$ and $B \vee D$ intersecting implies $A \vee D$ and $B \vee C$ intersecting.



Symmetric considerations yield that any one of the three intersections imply the existence of the remaining two. Thus, Vemblen's axiom states that any sufficiently nondegenerate 4-tuple of points in a projective plane forms a complete quadrangle.

5 Fano's axiom

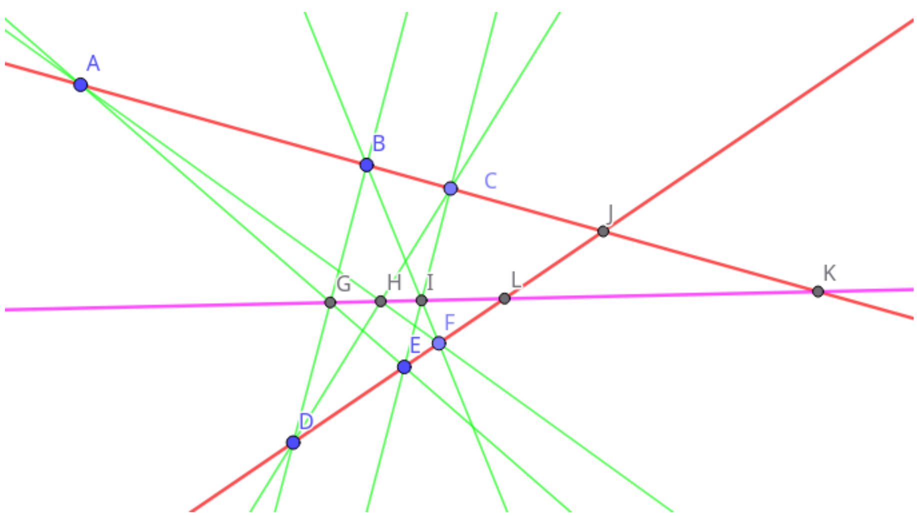
The following axiom is true for every projective plane except the configuration displayed here:



Theorem 5.1: Fano's axiom

The diagonal points of a complete quadrangle are never collinear.

6 Pappus' theorem



Theorem 6.1: Pappus' theorem

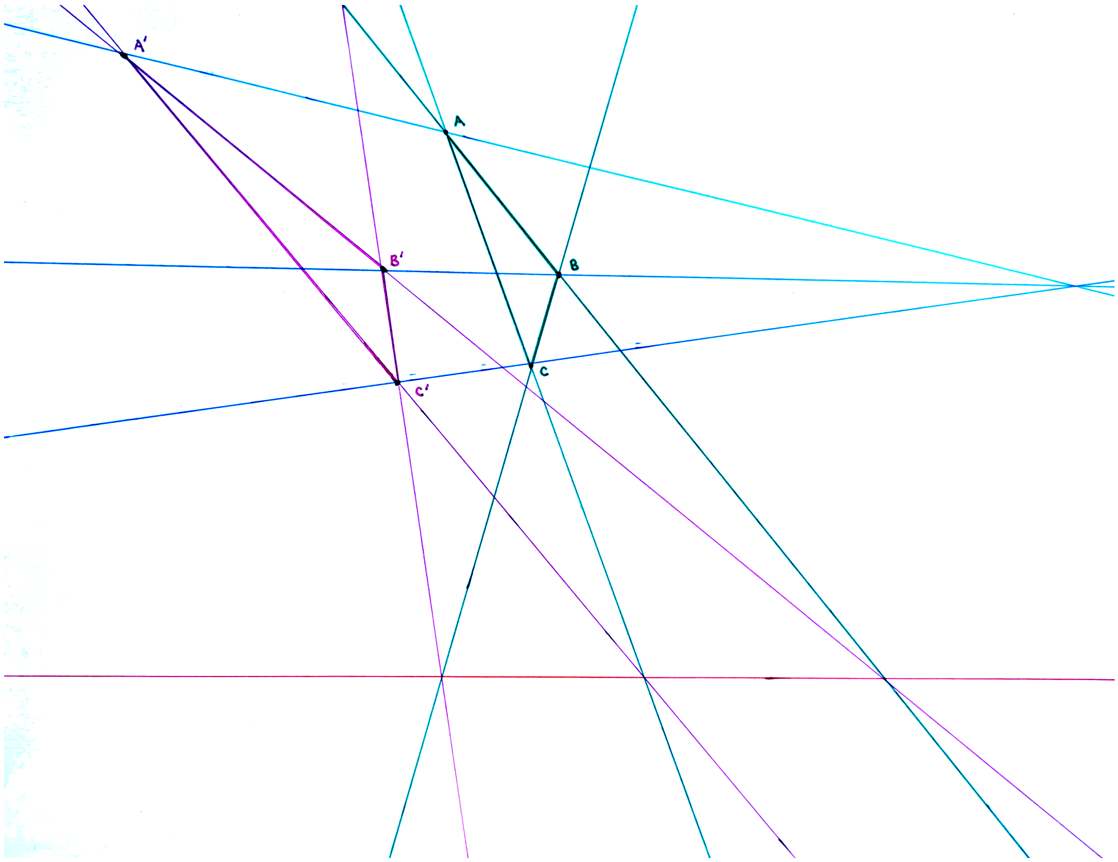
If $A, B, C, D, E, F \in \mathcal{P}$ are all distinct with A, B, C collinear and D, E, F collinear, then the points

$$G = (B \vee D) \wedge (A \vee E), \quad H = (A \vee F) \wedge (D \vee C), \quad I = (B \vee F) \wedge (C \vee E).$$

all exist and are collinear.

7 The big theorems of synthetic geometry

7.1 Desargues' theorem and its converse



7.2 Ceva's and Menalaus' theorems

7.3 Pascal's and Brianchon's theorems