Proposition 3.57 (Lebesgue Convering Lemma).

Let (X, d) *be a compact metric space and let* $\mathcal{O} = \{U_\iota\}_{\iota \in J}$ *be an open cover of* X.

Then $\mathcal{O}$ *admits a Lebesgue number.*

Proof of Proposition. Since X is compact, we may reduce to some finite subcover $\mathcal{O}_{\mathsf{fin}}$.

Let $x \in X$. Then $x \in U^\circ = U$ for some $U \in \mathcal{O}_{\mathsf{fin}}$. Since ∂U is compact (closed subset of $\overline{U}$ bounded by X),

$$\min_{p \in \partial U} d(x, p)$$

is well-defined (by compactness) and positive (since $x \in U^\circ$). Note, however, that there could be multiple $U_i \in \mathcal{O}_{\mathsf{fin}}$ containing x, so

$$\max_{U_i \ni x} \min_{p \in \partial U} d(x, p)$$

is also well-defined (since the max is over a finite set) and positive (max of positive set). But X is compact, so

$$m = \min_{x \in X} \max_{U_i \ni x} \min_{p \in \partial U} d(x, p)$$

is also well-defined (by compactness of X) and positive (min of positive set). Let $\delta = m/2$.

Now suppose $z \in Z \subseteq X$ where $\operatorname{diam} Z = \inf_{z, z' \in Z} d(z, z') < \delta$. Then for $z' \in Z$,

$$\begin{aligned} d(z, z') \leq \operatorname{diam} Z < \delta &= \frac{1}{2} \min_{x \in X} \max_{U_i \ni x} \min_{p \in \partial U} d(x, p) \\ &\leq \frac{1}{2} \max_{U_i \ni z} \min_{p \in \partial U} d(z, p) \\ &= \frac{1}{2} \min_{p \in \partial U_{\mathsf{max}}} d(z, p) \end{aligned}$$

for some $U_{\mathsf{max}} \in \mathcal{O}_{\mathsf{fin}}$. Since everything defined in sight has been continuous, U_{max} doesn't jump around (though it may be nonunique). The above inequality asserts that any point z' away from z doesn't venture outside the boundary of some U_{max}. Hence $Z \subseteq U_{\mathsf{max}}$, which completes the proof. ■