

The Cauchy-Schwarz Inequality

Justin T. Chun, 28 Sep 2026

For V a $\mathbb{C}$ -inner product space and $u, v \in V$, we have

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle$$

with equality exactly when u and v are linearly dependent.

Preliminary simplifiers. If u or v is zero, then u and v are linearly dependent and both sides equal zero, so we may assume u and v to be nonzero. Further, if u and v are nonzero but still linearly dependent, then there exists $\lambda \in \mathbb{C}$ such that $u = \lambda v$. Then

$$|\langle u, v \rangle|^2 = \langle u, v \rangle \langle v, u \rangle = \lambda \langle v, v \rangle \langle v, u \rangle = \langle v, v \rangle \langle u, u \rangle$$

which shows equality holds when u and v are linearly dependent. Further still, if Cauchy-Schwarz holds for all linearly independent unit vectors, then it holds for all linearly independent vectors: let a and b be corresponding unit vectors to u and v . Then

$$\begin{aligned} |\langle u, v \rangle|^2 &= \langle u, v \rangle \langle v, u \rangle = \langle a, b \rangle \langle b, a \rangle \langle u, u \rangle \langle v, v \rangle \\ &= |\langle a, b \rangle|^2 \langle u, u \rangle \langle v, v \rangle \\ &< \langle a, a \rangle \langle b, b \rangle \langle u, u \rangle \langle v, v \rangle = \langle u, u \rangle \langle v, v \rangle. \end{aligned}$$

So it suffices to show that for linearly independent unit vectors $u, v \in V$, that

$$|\langle u, v \rangle|^2 < 1.$$

Proof of Cauchy-Schwarz. Let V be a $\mathbb{C}$ -inner product space and let $u, v \in V$ be linearly independent unit vectors.

Then by linear independence, there is no $\lambda \in \mathbb{C}$ such that $u = \lambda v$.

So for all $\lambda \in \mathbb{C}$, we have $u \neq \lambda v$, hence $u - \lambda v \neq 0$, hence

$$\langle u - \lambda v, u - \lambda v \rangle > 0.$$

Set $\lambda = \langle u, v \rangle$. Then

$$\begin{aligned} 0 &< \langle u - \lambda v, u - \lambda v \rangle \\ &= \langle u, u \rangle - \lambda^* \langle u, v \rangle - \lambda \langle v, u \rangle + |\lambda|^2 \langle v, v \rangle = 1 - |\lambda|^2 \end{aligned}$$

so $|\langle u, v \rangle|^2 < 1$, as desired. ■