The Cauchy-Schwarz and Triangle Inequalities

Here we prove the Cauchy-Schwarz inequality using a geometric argument discovered by Tarung Bhimnathwala in 2021. We also give the standard proof of the triangle inequality.

1 The Cauchy-Schwarz Inequality

For vectors $|a\rangle$ and $|b\rangle$ in a complex inner product space, we have

$$|\langle a|b\rangle| \leq |a||b|$$

with equality holding when $a = \lambda b$ for some $\lambda \in \mathbf{C}$.

Proof. First we normalize everything. Define

$$|u\rangle = \frac{|a\rangle}{|a|}, \qquad |v\rangle = \frac{|b\rangle}{|b|}, \qquad \gamma = \frac{\langle u|v\rangle^*}{|\langle u|v\rangle|}$$

so that if any of the denominators are 0 then the result follows immediately.

Next we pick a nice basis. Let

$$|p\rangle = \frac{|u\rangle + |\gamma v\rangle}{2}, \qquad |q\rangle = \frac{|u\rangle - |\gamma v\rangle}{2}.$$

Note that

$$\begin{aligned}
\langle p|q\rangle &= \frac{1}{4}\langle u + \gamma v|u - \gamma v\rangle \\
&= \frac{1}{4}\left(\langle u|u\rangle - \langle u|\gamma v\rangle + \langle \gamma v|u\rangle - \langle \gamma v|\gamma v\rangle\right) \\
&= \frac{1}{4}\left(|u|^2 - 2\,\mathrm{Im}(\langle u|\gamma v\rangle) - |\gamma|^2|v|^2\right) \\
&= -\frac{1}{2}\,\mathrm{Im}(\gamma\langle u|v\rangle) = -\frac{1}{2}\,\mathrm{Im}(|\langle u|v\rangle|) = 0.
\end{aligned}$$

Also note that

$$\begin{aligned}
|p|^2 + |q|^2 &= \langle p|p\rangle + \langle q|q\rangle \\
&= \frac{1}{4}\left(\langle u + \gamma v|u + \gamma v\rangle + \langle u - \gamma v|u - \gamma v\rangle\right) \\
&= \frac{1}{2}\left(|u|^2 + |\gamma|^2|v|^2\right) = 1.
\end{aligned}$$

Now,

$$
\begin{aligned}
|\langle u|v\rangle| &= |\langle p+q|\gamma^*(p-q)\rangle| = |\gamma|\,|\langle p+q|p-q\rangle| \\
&= \left||p|^2 - |q|^2\right| = |1-2|q|^2| \leq 1.
\end{aligned}
$$

Thus, $|\langle a|b\rangle| \leq |a||b|$, as desired.

Note that equality holds when $|q|$ is either 0 or 1.

- If $|q| = 0$, then $|u\rangle = \gamma|v\rangle$.
- If $|q| = 1$, then since $|p|^2 + |q|^2 = 1$, we have $|p| = 0$, hence $|u\rangle = -\gamma|v\rangle$.

Thus, equality holds whenever $|a\rangle$ is a scalar multiple of $|b\rangle$. □

2 The Triangle Inequality

For vectors $|a\rangle$ and $|b\rangle$ in a complex inner product space, we have

$$|a+b| \leq |a| + |b|.$$

Proof. First, recall that for any complex number z, we have

$$(\operatorname{Re} z)^2 \leq (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 = |z|^2,$$

and thus $\operatorname{Re} z \leq |z|$. Now, note that

$$
\begin{aligned}
|a+b|^2 &= \langle a+b|a+b\rangle \\
&= |a|^2 + 2\operatorname{Re}(\langle a|b\rangle) + |b|^2 \\
&\leq |a|^2 + 2|\langle a|b\rangle| + |b|^2 \\
&\leq |a|^2 + 2|a||b| + |b|^2 \\
&= (|a| + |b|)^2,
\end{aligned}
$$

so that the desired result follows.

In order to have equality, $|a\rangle$ and $|b\rangle$ must satisfy the equality condition of the Cauchy-Schwarz inequality, i.e. $|a\rangle = \lambda|b\rangle$ with $\lambda \in \mathbf{C}$. But we also need $\langle a|b\rangle = \overline{\lambda}\langle b|b\rangle$ to be positive real, which forces $\lambda \in \mathbf{R}^+$. □