Coordinates in 3-space

Abstract

We derive the classical vector calculus identities using the language of differential forms, with which we assume some familiarity. The notation convention is slightly idiosyncratic:

- x and y are as they are in two dimensions,
- r is the xy-radius whereas ρ is the xyz-radius,
- ϕ is the vertical angle whereas θ is the horizontal angle.

The boxed equations in this writeup include the gradient, divergence, curl, and laplacian in cartesian, cylindrical, and spherical coordinates.

1 Coordinates of a Vector

Let $v \in \mathbf{R}^3$. We'll denote basis vectors by e_ι where $\iota \in \{x, y, z\}$.

Here, $\operatorname{atan}_2(a, b)$ is essentially $\arctan(\frac{a}{b})$, but adjusted to avoid undesirable divide-by-zero errors.

1.1 Cartesian

Thus we have for the vector v: first in cartesian, then from cylindrical, then from spherical.[1]

$$\begin{aligned} v &= v_x e_x + v_y e_y + v_z e_z \\ &= (v_r \cos v_\theta) e_x + (v_r \sin v_\theta) e_y + v_z e_z \\ &= (v_\rho \sin v_\phi \cos v_\theta) e_x + (v_\rho \sin v_\phi \sin v_\theta) e_y + (v_\rho \cos v_\phi) e_z \end{aligned}$$

1.2 Cylindrical

Now we have v first in cylindrical, then from spherical, then from cartesian.

$$\begin{aligned} v &= v_r e_r + v_\theta e_\theta + v_z e_z \\ &= (v_\rho \sin v_\phi) e_r + v_\theta e_\theta + (v_\rho \cos v_\phi) e_z \\ &= \left(\sqrt{v_x^2 + v_y^2}\right) e_r + (\operatorname{atan}_2(v_y, v_x)) e_\theta + v_z e_z \end{aligned}$$

1.3 Spherical

Now we have v first in spherical, then from cartesian, then from cylindrical.

$$\begin{aligned} v &= v_\rho e_\rho + v_\theta e_\theta + v_\phi e_\phi \\ &= \left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right) e_\rho + (\operatorname{atan}_2(v_y, v_x))\, e_\theta + \left(\arccos\left(\frac{v_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}}\right)\right) e_\phi \\ &= \left(\sqrt{v_r^2 + v_z^2}\right) e_\rho + v_\theta e_\theta + \left(\arccos\left(\frac{v_z}{\sqrt{v_r^2 + v_z^2}}\right)\right) e_\phi \end{aligned}$$

[1] Throughout, we'll state the cartesian case as a means of framing what these objects look like in coordinates "by default."

2 Vector Differentials in Tensor Notation

Here is where all the geometric intuition is hiding: these formulas describe the diagrams often found in derivations of these vector calculus identities. A fun exercise is to try recovering the diagrams.

2.1 Cartesian

The displacement differential is a straightforward sum of the basis forms tensored with their respective vectors. The surface area differential is a little trickier, but you can think of summing the areas of the xy, yz, and zx 2-forms and then tensoring with the basis vectors. The volume differential is the x displacement times the y displacement times the z displacement.

$$\begin{aligned} d\ell &= dx \otimes e_x + dy \otimes e_y + dz \otimes e_z \\ dS &= dy \wedge dz \otimes e_x + dz \wedge dx \otimes e_y + dx \wedge dy \otimes e_z \\ dV &= dx \wedge dy \wedge dz \otimes 1 = dx \wedge dy \wedge dz \end{aligned}$$

2.2 Cylindrical

The novelty is that the coefficent of $d\theta \otimes e_\theta$ is no longer just 1, but rather r. Thus, in every 2-form containing $d\theta$, the coefficent is also r, which is why the surface area form is like so. Similarly, the volume form has only one instance of $d\theta$, so the coefficient is just r.

$$\begin{aligned} d\ell &= dr \otimes e_r + r\, d\theta \otimes e_\theta + dz \otimes e_z \\ dS &= r\, d\theta \wedge dz \otimes e_r + dz \wedge dr \otimes e_\theta + r\, dr \wedge d\theta \otimes e_z \\ dV &= dr \wedge (r\, d\theta) \wedge dz \otimes 1 = r\, dr \wedge d\theta \wedge dz \end{aligned}$$

2.3 Spherical

Now things get more involved. The displacement has coefficient r in the $d\theta \otimes e_\theta$ term, and ρ in the $d\phi \otimes e_\phi$ term. This affects the surface area form, which has 2-form coefficents that are the product of the 1-form coefficients. Similarly, the volume form has coefficient $\rho^2 \sin\phi$.

$$\begin{aligned} d\ell &= d\rho \otimes e_\rho + \rho \sin\phi\, d\theta \otimes e_\theta + \rho d\phi \otimes e_\phi \\ dS &= \rho^2 \sin\phi\, d\theta \wedge d\phi \otimes e_\rho + \rho d\phi \wedge d\rho \otimes e_\theta + \rho \sin\phi\, d\rho \wedge d\theta \otimes e_\phi \\ dV &= \rho^2 \sin\phi\, d\rho \wedge d\theta \wedge d\phi \otimes 1 = \rho^2 \sin\phi\, d\rho \wedge d\theta \wedge d\phi \end{aligned}$$

3 Gradient of a Scalar Field

Let $f = f(x, y, z) = f(r, \theta, z) = f(\rho, \theta, \phi)$ be a scalar field. In general, $\nabla f = (df)^\sharp$.

The sharp symbol is a *musical isomorphism*, it turns basis 1-forms into basis vectors.

It is closely related to the displacement differential in this context.

Basically, we compute the gradient by computing the sharp map of the exterior derivative of f.

3.1 Cartesian

The sharp $(\cdot)^\sharp$ maps $d\iota \mapsto e_\iota$ in Cartesian coordinates:

$$\boxed{\nabla f = (\partial_x f)e_x + (\partial_y f)e_y + (\partial_z f)e_z}$$

3.2 Cylindrical

In cylindrical coordinates, we have $(dr)^\sharp = e_r$, $(rd\theta)^\sharp = e_\theta$, $(dz)^\sharp = e_z$, which implies:

$$\boxed{\nabla f = (\partial_r f)e_r + (r^{-1}\partial_\theta f)e_\theta + (\partial_z f)e_z}$$

3.3 Spherical

In spherical coordinates, we have $(d\rho)^\sharp = e_\rho$, $(\rho \sin\phi d\theta)^\sharp = e_\theta$, $(\rho d\phi)^\sharp = e_\phi$, which implies:

$$\boxed{\nabla f = (\partial_\rho f)e_\rho + ((\rho \sin\theta)^{-1}\partial_\theta f)e_\theta + (\rho^{-1}\partial_\phi f)e_\phi}$$

4 Divergence of a Vector Field

Let $F = F_x(x, y, z)e_x + F_y(x, y, z)e_y + F_z(x, y, z)e_z$ be a vector field. In general, $\nabla \cdot F = \star d \star (F^\flat)$.

In order to compute the Hodge star, it is necessary to know the scaling factor of the volume form dV.

Let's call this scaling factor μ.

4.1 Cartesian

Here $\mu = 1$, i.e. $dV = dx \wedge dy \wedge dz$. The flat $(\cdot)^\flat$ maps $e_\iota \mapsto d\iota$ in Cartesian coordinates:

$$\begin{aligned} F^\flat &= F_x\, dx + F_y\, dy + F_z\, dz \\ \star F^\flat &= F_x\, dy \wedge dz + F_y\, dz \wedge dx + F_z\, dx \wedge dy \\ d \star F^\flat &= (\partial_x F_x + \partial_y F_y + \partial_z F_z)\, dx \wedge dy \wedge dz \end{aligned}$$

Thus,

$$\boxed{\nabla \cdot F = \partial_x F_x + \partial_y F_y + \partial_z F_z}$$

4.2 Cylindrical

Here $\mu = r$, i.e. $dV = r(dr \wedge d\theta \wedge dz)$. In cylindrical coordinates, we have:

$$\begin{aligned} F^\flat &= F_r\, dr + rF_\theta\, d\theta + F_z\, dz \\ \star(F^\flat) &= (rF_r) d\theta \wedge dz + F_\theta\, dz \wedge dr + (rF_z) dr \wedge d\theta \\ d \star (F^\flat) &= (\partial_r(rF_r) + \partial_\theta F_\theta + r\partial_z F_z) dr \wedge d\theta \wedge dz \end{aligned}$$

$$\boxed{\nabla \cdot F = r^{-1}\left(\partial_r(rF_r) + \partial_\theta F_\theta\right) + \partial_z F_z}$$

4.3 Spherical

Here $\mu = \rho^2 \sin\phi$, i.e. $dV = \rho^2 \sin\phi(d\rho \wedge d\theta \wedge d\phi)$. In spherical coordinates, we have:

$$\begin{aligned} F^\flat &= F_\rho\, d\rho + \rho \sin\phi F_\theta\, d\theta + \rho F_\phi\, d\phi \\ \star(F^\flat) &= \left(\rho^2 \sin\phi F_\rho\right)\, d\theta \wedge d\phi + (\rho F_\theta)\, d\phi \wedge d\rho + (\rho \sin\phi F_\phi)\, d\rho \wedge d\theta \\ d \star (F^\flat) &= \left(\sin\phi \partial_\rho(\rho^2 F_\rho) + \rho\partial_\theta F_\theta + \rho\partial_\phi(\sin\phi F_\phi)\right) d\rho \wedge d\theta \wedge d\phi \end{aligned}$$

$$\boxed{\nabla \cdot F = \rho^{-2}\partial(\rho^2 F_\rho) + (\rho \sin\phi)^{-1}\left(\partial_\theta F_\theta + \partial_\phi(\sin\phi F_\phi)\right)}$$

5 Curl of a Vector Field

Let F be as before. In general, $\nabla \times F = (\star dF^\flat)^\sharp$.

5.1 Cartesian

We compute:

$$\begin{aligned} F^\flat &= F_x\,dx + F_y\,dy + F_z\,dz \\ dF^\flat &= (\partial_y F_z - \partial_z F_y)\,dy \wedge dz + (\partial_z F_x - \partial_x F_z)\,dz \wedge dx + (\partial_x F_y - \partial_y F_x)\,dx \wedge dy \\ \star dF^\flat &= (\partial_y F_z - \partial_z F_y)\,dx + (\partial_z F_x - \partial_x F_z)\,dy + (\partial_x F_y - \partial_y F_x)\,dz \end{aligned}$$

Thus,

$$\boxed{\nabla \times F = (\partial_y F_z - \partial_z F_y)e_x + (\partial_z F_x - \partial_x F_z)e_y + (\partial_x F_y - \partial_y F_x)e_z}$$

5.2 Cylindrical

Recall $\mu = r$.

We compute:

$$\begin{aligned} F^\flat &= F_r\,dr + rF_\theta\,d\theta + F_z\,dz \\ dF^\flat &= (\partial_\theta F_z - r\partial_z F_\theta)\;d\theta \wedge dz + (\partial_z F_r - \partial_r F_z)\;dz \wedge dr + (\partial_r(rF_\theta) - \partial_\theta F_r)\;dr \wedge d\theta \\ \star dF^\flat &= \left(r^{-1}\partial_\theta F_z - \partial_z F_\theta\right) dr + r\,(\partial_z F_r - \partial_r F_z)\,d\theta + r^{-1}\,(\partial_r(rF_\theta) - \partial_\theta F_r)\,dz \end{aligned}$$

Thus,

$$\boxed{\nabla \times F = \left(r^{-1}\partial_\theta F_z - \partial_z F_\theta\right) e_r + (\partial_z F_r - \partial_r F_z)\,e_\theta + r^{-1}\,(\partial_r(rF_\theta) - \partial_\theta F_r)\,e_z}$$

5.3 Spherical

Recall $\mu = \rho^2 \sin\phi$.

We compute:

$$\begin{aligned} F^\flat &= F_\rho\,d\rho + \rho\sin\phi F_\theta\,d\theta + \rho F_\phi\,d\phi \\ dF^\flat &= (\rho\sin\phi)^{-1}\,(\partial_\phi(\sin\phi F_\theta) - \partial_\theta F_\phi)\,(\rho\sin\phi d\theta) \wedge (\rho\,d\phi) \\ &\quad + \rho^{-1}\,(\partial_\rho(\rho F_\phi) - \partial_\phi F_\rho)\,(\rho\,d\phi) \wedge d\rho \\ &\quad + (\rho\sin\phi)^{-1}\,(\partial_\theta F_\rho - \sin\phi\partial_\rho(\rho F_\theta))\,d\rho \wedge (\rho\sin\phi d\theta) \\ \star dF^\flat &= (\rho\sin\phi)^{-1}\,(\partial_\phi(\sin\phi F_\theta) - \partial_\theta F_\phi)\,d\rho \\ &\quad + (\partial_\rho(\rho F_\phi) - \partial_\phi F_\rho)\sin\phi\,d\theta \\ &\quad + (\rho\sin\phi)^{-1}\,(\partial_\theta F_\rho - \sin\phi\partial_\rho(\rho F_\theta))\,(\rho\,d\phi) \end{aligned}$$

Thus,

$$\boxed{\nabla \times F = (\rho\sin\phi)^{-1}\,(\partial_\phi(\sin\phi F_\theta) - \partial_\theta F_\phi)\,e_\rho + \rho^{-1}\,(\partial_\rho(\rho F_\phi) - \partial_\phi F_\rho)\,e_\theta + \rho^{-1}\,\left((\sin\phi)^{-1}\partial_\theta F_\rho - \partial_\rho(\rho F_\theta)\right) e_\phi}$$

6 Laplacian of a Scalar Field

Let f be as before. In general, $\nabla^2 f = \star d \star df$.

6.1 Cartesian

We compute:

$$\begin{aligned} df &= \partial_x f\,dx + \partial_y f\,dy + \partial_z f\,dz \\ \star df &= \partial_x f\,dy \wedge dz + \partial_y f\,dz \wedge dx + \partial_z f\,dx \wedge dy \\ d \star df &= (\partial_{xx} f + \partial_{yy} f + \partial_{zz} f)\,dx \wedge dy \wedge dz \end{aligned}$$

Thus,

$$\boxed{\nabla^2 f = \partial_{xx} f + \partial_{yy} f + \partial_{zz} f}$$

6.2 Cylindrical

We compute:

$$\begin{aligned} df &= \partial_r f\,dr + r^{-1}\partial_\theta f\,(r\,d\theta) + \partial_z f\,dz \\ \star df &= \partial_r f\,(r\,d\theta) \wedge dz + r^{-1}\partial_\theta f\,dz \wedge dr + \partial_z f\,dr \wedge (r\,d\theta) \\ d \star df &= \left(r^{-1}\partial_r(r\partial_r f) + r^{-2}\partial_{\theta\theta} f + \partial_{zz} f\right) dr \wedge (r\,d\theta) \wedge dz \end{aligned}$$

Thus,

$$\boxed{\nabla^2 f = r^{-1}\partial_r(r\partial_r f) + r^{-2}\partial_{\theta\theta} f + \partial_{zz} f}$$

6.3 Spherical

We compute:

$$\begin{aligned} df &= \partial_\rho f\,d\rho + (\rho\sin\phi)^{-1}\partial_\theta f\,(\rho\sin\phi\,d\theta) + \rho^{-1}\partial_\phi f\,(\rho\,d\phi) \\ \star df &= \partial_\rho f(\rho\sin\phi\,d\theta) \wedge (\rho\,d\phi) + (\rho\sin\phi)^{-1}\partial_\theta f(\rho\,d\phi) \wedge d\rho + \rho^{-1}\partial_\phi f(d\rho \wedge (\rho\sin\phi\,d\theta)) \\ d \star df &= \left(\rho^{-2}\partial_\rho(\rho^2\partial_\rho f) + (\rho\sin\phi)^{-2}\,\partial_{\theta\theta} f + \left(\rho^2\sin\phi\right)^{-1}\partial_\phi(\sin\phi\,\partial_\phi f)\right) d\rho \wedge (\rho\sin\phi\,d\theta) \wedge (\rho\,d\phi) \end{aligned}$$

Thus,

$$\boxed{\nabla^2 f = \rho^{-2}\partial_\rho(\rho^2\partial_\rho f) + (\rho\sin\phi)^{-2}\partial_{\theta\theta} f + (\rho^2\sin\phi)^{-1}\partial_\phi(\sin\phi\,\partial_\phi f)}$$